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MACHINES AS TOOLS IN TEACHER EDUCATION[1]

The aim of this chapter is to present some issues concerning teacher education, at both primary and secondary school levels, drawing on the activity of the Laboratory of Mathematical Machines at the Department of Mathematics of the University of Modena and Reggio Emilia (MMLab: www.mmlab.unimore.it). After having defined what are mathematical machines (geometrical and arithmetical machines, as well), we shall illustrate shortly the theoretical framework of semiotic mediation after Vygotsky, where the activity for prospective and practising school teachers is situated. We shall offer two examples. The first concerns arithmetical machines related to the meaning of place value in primary school and the second geometrical machines related to the meaning of axial symmetry in secondary school. Activity takes place in small size (25-30 students) laboratory settings for prospective and practising school teachers, according to the Italian standards for teacher education and to the implementation realized at the Faculty of Education of the University of Modena and Reggio Emilia.

INTRODUCTION

The Laboratory of Mathematical Machines of the University of Modena and Reggio Emilia is a well known research centre for the teaching and learning of mathematics by means of instruments (Maschietto, 2005; Larousserie, 2005). The name comes from the most important collection of the Laboratory, containing more than 200 working reconstructions (based on the original sources) of many mathematical instruments taken from the history of geometry. Briefly,

> a mathematical (or, better, a geometrical) machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law.

Examples are the standard compass (that forces a point to go on a circular trajectory, see below) and the Dürer glass used as a perspectograph (that transform a point into its perspective image on a glass from a given point). However, the activities in the MMLab are not limited to the above kind of instruments. Also activities with physical machines concerning arithmetics are carried out. For brevity, we shall call them "arithmetical machines". Briefly,

> an arithmetical machine is a tool that allows the user to perform at least one of the following actions: counting; making calculations; representing numbers.

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Tools from ICT (Information and Communication Technologies, e.g. calculators, dynamic software) are available and frequently used in the MMLab, but, in this chapter they will not be considered. Rather, we shall focus on artefacts which, because of their very origin and the concrete constructions, foster both historical-cultural and manipulative approaches to mathematics.

Using effectively such artefacts in the mathematics classroom is a true challenges for teacher, as specific professional competences, which cannot be taken for granted, are required,. The complexity of these competences is consistent with the multidimensional feature of *mathematical knowledge for teaching* (Ball et al., submitted).

In all the research studies carried out by the team of the MMLab, at least three analytical components are present:

– an *epistemological component*, with attention to mathematical meaning;
– a *didactical component*, with attention to the classroom processes;
– a *cognitive component*, with attention to processes of learning (Arzarello and Bartolini Bussi, 1998).

This approach for activity in primary and secondary classrooms is carried out also in prospective and practising teacher education that takes place within the Faculty of Education at Reggio Emilia. According to the Italian governmental regulations issued in 1998, teacher education is organized around three main kinds of activities: *lectures* (for large groups of prospective teachers, up to 100 and more), *in-school apprenticeship* (individual participation in standard classroom activities, under the supervision of expert teachers) and *laboratories*.

The activities described in this chapter take place in the laboratory settings, which are the same size as a standard classroom (25-30 participants). In most mathematical laboratories, in our faculty, prospective teachers explore geometrical and arithmetical machines[2]. At the beginning, the teacher educator acts as the classroom teacher, whilst the prospective teachers act as the students: usually they are given tasks that are similar to the ones that could be used with primary and secondary students. Later metacognitive activity takes place, to make explicit the links between the mathematical activity, as experienced by the prospective teachers, and the theoretical framework. In this way the very tasks acquire a paradigmatic feature that allows the prospective teachers to give sense to the theoretical framework. We shall come back again to the differences between the activity with students and with prospective teachers in the conclusions.

In the following section, some elements of the theoretical framework will be summarized very briefly, drawing on the chapter by Bartolini Bussi and Mariotti (in press), before discussing some cases. In the mentioned framework, the artefacts of MMLab are interpreted *as tools of semiotic mediation for the construction of mathematical meanings under the teacher's guide.* A Vygotskian framework is particularly suitable, because of the importance of the teacher's role and the focus on both the concreteness (that requires direct manipulation) and on the explicit historical reference of the artefacts.

THEORETICAL FRAMEWORK: AN OUTLINE.

Artefacts

The word *artefact* is generally used in a very general way and encompasses oral and written forms of language, texts, physical tools used in the history of arithmetic (e.g., abaci and mechanical calculators) and geometry (e.g., straightedge and compass), tools from ICT, manipulatives, and so on. According to Rabardel (1995), an *artefact* is a material or symbolic object per se. The *instrument* (to be distinguished from the artefact) is defined as a hybrid entity made up of both artefact-type components and schematic components that are called *utilization schemes*. The utilization schemes are progressively elaborated when an artefact is used to accomplish a particular task; thus the instrument is a construction of an individual[3]. It has a psychological character and it is strictly related to the context within which it originates and its development occurs. The elaboration and evolution of the instruments is a long and complex process that Rabardel names *instrumental genesis*. Instrumental genesis can be articulated into two coordinated processes: *instrumentalisation*, concerning the emergence and the evolution of the different components of the artefact, drawing on the progressive recognition of its potentialities and constraints; *instrumentation*, concerning the emergence and development of the utilization schemes. "In the instrumentation process, the subject develops, while in the instrumentalization process, it is the artefact that evolves" (Rabardel, 1995, p. 12)[4]. In the following, we shall illustrate how both processes, that have been studied by Rabardel in cognitive ergonomy, apply to classroom activity.

The artefacts used in the MMLab and selected for this chapter are machines concerning geometry and arithmetic. Unlike some artefacts from ICT, they are to be concretely handled; they require motor abilities; they put up resistance to motion; and they need time to be explored. One might observe that using concrete manipulatives to teach mathematics is a long-established educational strategy, at least with young learners, based on theories claiming that children need concrete referents to develop abstract mathematics concepts (Piaget, 1966). This assumption has been often supported by the implicit or explicit claim that educational manipulatives are "transparent" for mathematical meanings (see also Chapter 7, this volume). We take the distance from this view in relation to the following two different issues:

– *The learners' age:* we claim that concrete manipulatives are to be used not only with children but also with older students, up to the tertiary level; we shall offer examples showing that some very sophisticated mathematical processes (e.g. the elaboration of definitions and the construction of proofs) can take advantage of a guided manipulation of concrete artefacts at all ages;

– *The transparency of the artefact*: we claim that artefacts are not transparent, rather they "become efficient, relevant, and transparent through their use in specific activities and in relation to the transformations that they undergo in the hands of users" (Meira, 1998). To Meira, transparency (if any) is not an inherent (objective) feature of the tool, but emerges through the very use of the tool itself. The artefacts used in the MMLab seem "transparent", as unlike the tools of the ICT, the functioning is completely accessible and there is no hidden engine or software inside. Yet, in spite of this functioning transparency, they are not "transparent" where mathematical meaning is concerned.

Cultural artefacts play an essential role in the Vygotskian approach. Vygotsky pointed out that, in the practical sphere, human beings use technological or concrete tools, reaching achievements that would otherwise have remained out of reach. In mental activities, human beings reach higher levels through mediation by artificial stimuli (signs or semiotic tools), that are referred to as psychological tools. In most of the further literature signs have been interpreted as linguistic signs, due to the greater importance attached by Vygotsky to language. Yet, Vygotsky (1981) himself suggested other examples among which there are various systems for counting and mechanical drawings.

Example: the Compass

A simple, yet meaningful example for mechanical drawing, is given by the tool evoked by Hero in his mechanical dynamic procedural definition of circle, as

> the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position (Heath, 1908, p.184).

This tool (Figure 1) is a different version of the pair of compasses (Figure 2). In the former, the straight line is materialized by the piece of the bar (OC) between the two hands, whilst in the latter is given by the (not visible) base (OP) of the triangle formed by the legs.

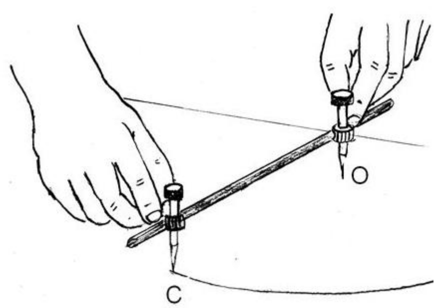

Figure 1. Beam compass

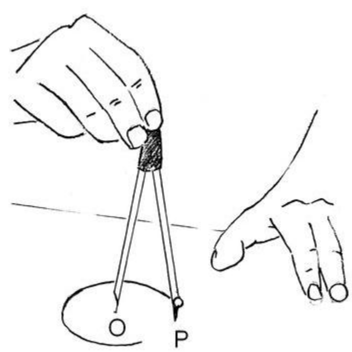

Figure 2. Pair of compasses

As *technical tools*, both a beam compass and a pair of compasses (we shall use in both cases the word compass, for brevity) are used to produce round shapes: the ways of handling, evoked in the above figures, are different and are neither spontaneous nor simple, especially for young pupils. As a *psychological tool* either has the potentiality to evoke the peculiar feature of circles (i.e., the constancy of the radius) and to create the link with the geometrical static relational definition of Euclid:

> A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another (Heath, 1908, p. 183).

As a technical tool it is *externally* oriented; as a psychological tool it is *internally* oriented (Vygotsky, 1978, p.55). A compass may be used to produce a solution of the following construction problem, from Euclid's elements Book 1 (see, Heath, 1908 p.241).

> To construct an equilateral triangle on a given finite straight line.

In the proof, no compass is mentioned. Rather, among others, the third postulate is recalled:

> Let the following be postulated: To describe a circle with any centre and radius.

In other words, what is important is not the very drawing of the circle carried out with some artefact, but the possibility to describe it and to use its peculiar properties. The original proof follows.

> Proposition 1.
> To construct an equilateral triangle on a given finite straight line.
> Let AB be the given finite straight line.
> Thus it is required to construct an equilateral triangle on the straight line AB.
> With centre A and distance AB, let the circle BCD be described [post. 3];
> Again, with centre B and distance BA let the circle ACE be described [post. 3].
> And from the point C in which the circles cut one another, to the points A, B, let the straight lines CA, CB be joined [post. 1].
> Now, since the point A is the centre of the circle CDB, AC is equal to AB [def. 15].
> Again, since the point B is the centre of the circle CAE, BC is equal to BA [def. 15].
> But CA was also proved equal to AB, therefore each of the straight lines CA and BC is equal to AB.
> And things which equal the same thing also equalto one another [C.N. 1].
> Therefore CA is also equal to BC.
> Therefore the three straight lines CA, AB, and BC are equal to one another.
> Therefore the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB.
> (Being) what it was required to do[5].

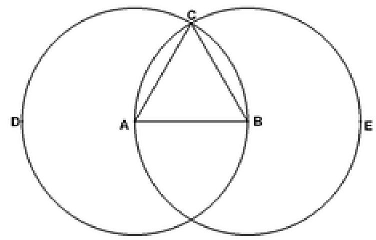

Figure 3. Euclid's Proposition 1: The drawing

In technical drawing lessons, secondary school students are taught a solution of the same problem (Figure 4), where two small signs are traced by means of the compass to find the third vertex of the triangle. They might be able to carry out the concrete operations with the compass and to describe it carefully (*how*), without being able to give any geometrical justification (*why*); when this happens, the students are using the compass only as a technical tool to produce a drawing, but not (yet) as a psychological tool, because they are not (yet) aware that the solution draws on the property of circle to be the locus of points at a given distance from a given point.

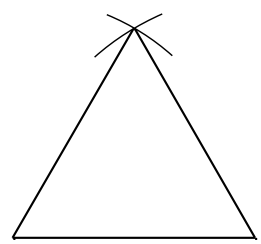

Figure 4. The construction of an equilateral triangle in technical drawing handbooks.

It might be considered only a first step in the construction of the meaning of circle, as soon as the students appropriate this meaning the above construction problem is not challenging any more and becomes a trivial exercise. The above discussion suggests a meaningful task to be introduced into teacher education, concerning the solution of a particular construction problem.

A task in teacher education. On a standard white (not squared) sheet of paper, two circles are drawn: the radii are 3 cm and 2 cm and the distance between the two centres is 7 cm. The problem is:

> *Draw a circle with a radius of 4 cm tangent to the given circles. You can use instruments. Explain clearly the method so that others can use it. Explain carefully why the method works*

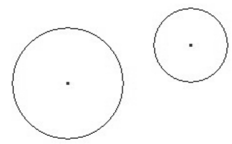

Figure 5. The drawing of the task

The task is twofold as the learner is explicitly asked to produce both a correct solution (*how*) and a geometrical justification of the solution (*why*). The artefact "compass" is evoked by the hint to use instruments: it has the potential of suggesting both a solution and a justification, that is, on the one hand, functionally linked to the task and, on the other hand, explicitly related to mathematics knowledge. However this potential is not trivial to be exploited by learners, regardless of age and expected school education . We have tested this problem with both young students from grade 5 (Bartolini Bussi et al. 2007) and prospective teachers, who have enrolled in the college for primary teacher education. In the latter case, one might expect solutions referring to argumentation (as prospective teachers have studied some geometry in secondary school); yet the early solutions given by undergraduates are similar to the solutions given by children. In most cases the compass is used only to draw the new circle (with a 4 cm radius), and not to find the centre of that circle. Rather, the centre is found by trial and error, evidence is given by the many small holes that appear by transparency in the sheet of paper. Again, the compass is used as a technical tool and not as a psychological tool. The very formulation of the task (*how* and *why*) allows the teacher educator to raise some issues in the discussion of the solutions. When the solution is found, even by trial and error, it is easily recognized that the problem is equivalent to the problem of finding a triangle with given sides. Hence the position of C may be calculated by intersecting two circles (with radii (3 + 4) cm and (2 + 4) cm respectively, see the dotted circles in Figure 6). The justification for this solution (*why*) requires the student to call into play some particular features: the Euclidean definition of circle; the equality of the distance between the two centres of two (externally) tangent circles to the sum of the radii.

When either the primary school pupils (under the teacher's guidance) or the prospective teachers (under the teacher educator's guidance) become aware of the relationships between the definition of circle, the function of this definition in the solution of the problem and the use of the compass, the process of transformation of the compass into a psychological tool is started. The justification of the method assumes the form of a true mathematical proof, with explicit reference to the definition and to the property of tangent circles.

This process is neither spontaneous nor short. It is the responsibility of either the teacher or the teacher educator to guide this process, fostering the transformation of the students' texts (*situated* in the practical activity carried out with the artefact) into mathematical texts.

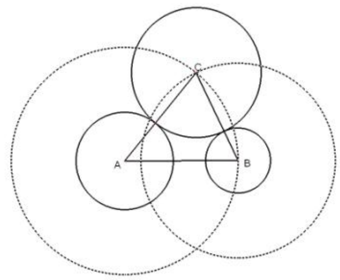

Figure 6. A solution by pair of compasses.

Semiotic Mediation

The process described above for the compass is similar to the process observed for other artefacts (Bartolini Bussi and Mariotti, in press) which can be linked, on the one hand, to meaningful tasks and, on the other hand, to some pieces of mathematics knowledge. In this paragraph we shall use the general term "teacher" to mean both the school teacher and the teacher educator and, similarly, the term "student" to mean both the pupil and the prospective teacher. The teacher, after having designed a meaningful task which refers to mathematics knowledge and may be solved by means of the compass (the left side of the diagram of the Figure 6), has the responsibility of observing and analysing the situated texts produced by the students and of designing and implementing their transformation into mathematical texts (the right part of the diagram).

The word "text" is used in a broad sense, to include not only written texts, but also gestures and gazes (that sometimes cannot be easily transformed into words by the students), drawings and whatever sign is used to make sense of and to communicate the procedure. The teacher's role in this process may be described as follows: *he/she uses the artefact as a tool of semiotic mediation*. For a detailed presentation of semiotic mediation in accordance with a Vygotskian approach, the reader might refer to Bartolini Bussi and Mariotti (in press). In short, one might say that the artefact is drawn by the teacher into the solving process both as a technical tool and as a psychological tool. In fact, on the one hand, it allows the user (either the teacher or the student) to produce a solution, and, on the other hand, it may evoke the cultural elaborations that are deposited on it from the time of Euclid (e. g., the peculiar features of circle). Hence the utilization schemes of students evolve and the they construct the meaning of circle as a locus of points in the same plane, equidistant from a given point, to the extent to be able to mobilize this piece of knowledge in problem solving.

In the Figure 7, the epistemological, the didactical and the cognitive components are articulated with each other. The *cognitive component* concerns the higher triangle " task – artefact – situated texts"; it is an evolutionary component, because, during time, the process of internalization in the zone of proximal development, enriches pupils' cognitive processes and changes the produced texts. The *epistemological component* concerns the left triangle "task - mathematics knowledge – mathematical texts". The *didactical component* concerns the right triangle "task, situated texts, mathematical texts", where teaching is in the foreground. In the last two a crucial function is played by the artefact, which, in this scheme, has been put in the centre.

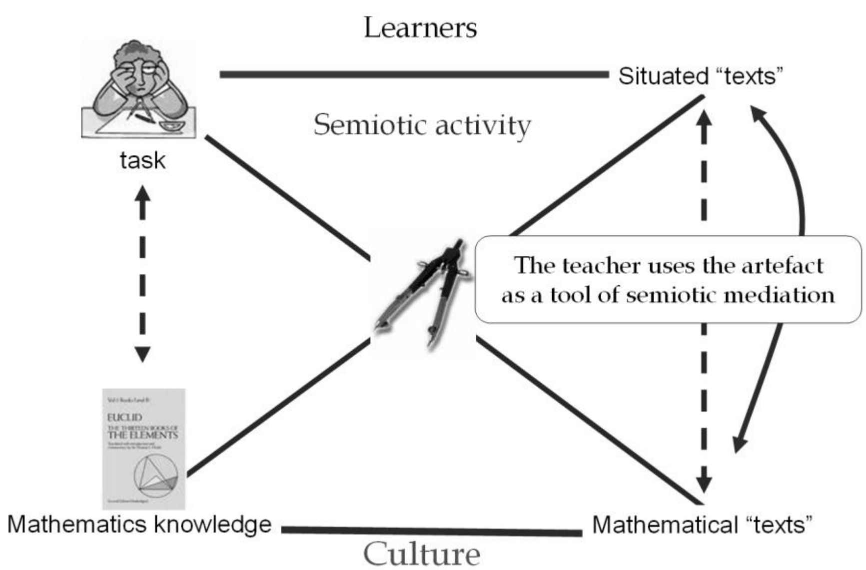

Figure 7. Semiotic mediation

In the following sections, other examples will be discussed. All the machines are taken from case studies developed with primary or secondary school students, before being applied to teacher education. The differences between classroom and teacher education settings will be addressed in the conclusions.

PRIMARY SCHOOL: PLACE VALUE FOR NATURAL NUMBERS IN BASE TEN

The Object to be Mediated

The representation of numbers is an immense field of research from the historical perspective (Menninger, 1958), from the anthropological perspective (Crump, 1992), and from the cognitive perspective (Tolchinsky, 2003). From a didactical perspective, the manipulation of written numbers and of operation algorithms is a general requirement in all the school systems, at least from primary school level on. The object to be mediated, in this case, is *the place value of digits:* this is an overarching meaning that is related to other meanings (e.g. one-to-one correspondence; numbers as operators) which will be mentioned below.

The Network of Artefacts

The worldwide interest in this topic and the age of the pupils involved has fostered the production of many manipulatives ("arithmetical machines"), which are still merchandised for and used in primary schools. We have selected a network of artefacts, which are documented in the history of mathematics. We shall briefly describe them below, before analysing the related instrumentalisation and instrumentation processes. We use the term *network of artefacts* to mean that no individual artefact is sufficient to form the meaning of place value to the extent of constructing the arithmetic operations and algorithms; rather it is the very system of them that can form this meaning in the plane of user's consciousness, together with the awareness of the different features of each artefact.

Counting sticks (dating back to ancient China) are thin bamboo or plastic sticks: the sticks are counted, grouped and bundled (and tied with ribbons or rubber bands) into tens for counting up to hundred; ten-bundles are grouped and bundled into hundreds and so on .

Figure 8. *Counting sticks*

The *spike abacus* consists of 3 spikes and 27 beads (or more). Each spike represents a particular position of a digit and can have a maximum of 9 beads. Another version (dating back to the Roman age) has no spike but grooves (the *grooved abacus*), where pebbles or other counters to represent numbers are placed.

The *pascaline*[6] is a mechanical calculator (see Figure 10), with a gear train (Maschietto and Ferri, in press). When the lower right wheel has turned a complete rotation, the upper right wheel makes the central wheel to go one step ahead. The same happens when the central wheel has turned a complete rotation: the upper left wheel makes the lower left wheel to go one step ahead. Digits from 0 to 9 are written on the lower wheels. The three small triangles on the bottom side point at a digit each, so that every 3 digit number is represented by the 3 pointed digits. The functioning is similar to the one of old mechanical odometers.

The above artefacts are modern reconstructions of traditional artefacts (Menninger, 1958); most were used for reckoning also before the place value of digits was established in writing. The explicit historical dimension differentiates them from other artefacts, such as the multibase blocks (Sriraman and Lesh, 2007) and the ICT tools. The teacher may benefit from introducing an historical discourse in the mathematics classroom, with even recourse to selected historical sources, to introduce the pupils into the flow of mathematics culture. This additional potential for the construction of mathematical meanings will not be explored in this chapter.

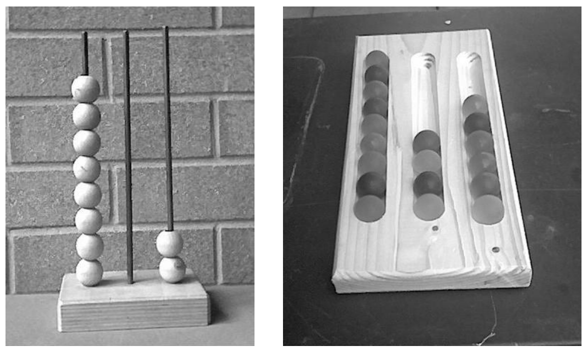

Figure 9. A spike (left) and a grooved (right) abacus

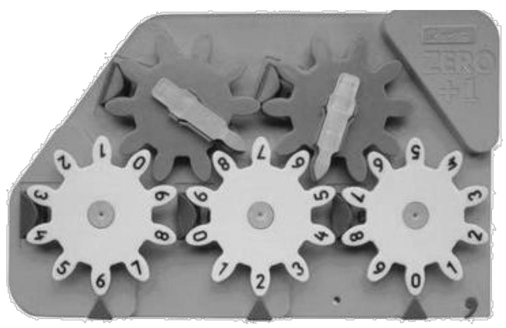

Figure 10. The pascaline "zero+1"

Meanings

It is beyond the scope of this chapter to analyse in detail the mathematical meanings potentially attached to the above artefacts (Bartolini Bussi and Mariotti, in press; Bartolini Bussi and Boni, submitted). In short:

- *One-to-one correspondence* is in the foreground for the counting sticks and abaci (although for the spike or grooved abaci, it works only for a very limited number of beads, i.e., 9 in the base ten representation, afterwards, conventions about grouping, composing and exchange need to be used).
- *Grouping* i.e. *composing (groups of ten)*is in the foreground for the counting sticks and abaci.

- *Number symbols* are written only on the pascaline. In the shift from counting sticks and abaci to written representations of numbers, zero appears as a place holder, whilst in the pascaline zero is rather a label.
- *The generation of the written number sequence* is in the foreground in the pascaline. It is generated by iterating the function "+1" concretized by one-tooth clockwise rotation of the right wheel. This is important when addition is at stake. With pascaline, the function of the two addenda is not symmetrical (the second one is an operator on the first one), whilst with counting sticks and abaci, the addition works as a binary operation. The above differences justify why it is not only better but necessary to refer to a network of artefacts rather than to a sole artefact: their complementarity is meaningful and suggests metacognitive tasks aimed at comparing their potential.

Instrumentalization

The difference between the artefacts (in spite of the similarity of meanings) makes this phase very important. Before the emergence of the evolution of the different components of the artefact, the given artefact is in the foreground. The quoted artefacts have some peculiar features: they consist of different parts with relationships with each other. A very simple example of instrumentalization is described by Bartolini Bussi and Boni (2003), concerning the spike abacus. In a mathematical discussion with second graders (7 year olds) about the function of abacus, the pupils had in front of them their personal abacus with four wires. The need for representing numbers beyond 9999 suggested the solution of placing side by side as many abaci as needed. The evolution of the artefact was progressive, from the juxtaposition of several abaci to the "mental" design of a new abacus with as many wires as needed.

Instrumentation

When a task is given, to be solved by means of one of the above artefacts, instrumentation is at work. Each individual learner constructs her/his own utilization schemes. A learner, while describing the process of using an artefact to solve a specific task, produces usually situated texts (see Figure 7), where metaphors and even gestures and gazes are very frequent and aim at conveying part of the meaning.

Examples of Tasks for Teacher Education.

A small group of prospective teachers was given one of the above arithmetical machines together with rulers for measuring, and paper and pencil.
Instrumentalisation tasks.

1. Produce a carefully written description of the artefact you have received, the parts, the size, the shapes, their spatial relationships, and so on. You can use words and drawings.
2. Design with everyday materials an arithmetical machine that can be used in the place of the one that has been offered you.
Instrumentation tasks
3. Represent the number 107 by means of counting sticks (or an abacus, or a pascaline). Describe carefully how you have realized the representation and why you are sure that it is correct.
4. Compare the utilization schemes in the above task, when different artefacts are into play.
5. Add 108 and 245 by means of counting sticks (or an abacus, or a pascaline). Describe carefully and justify your process.

In the spike or grooved abaci, before using the conventional term "exchange" (ten beads with one bead in a different position, hence with a different value) or "compose", pupils and even prospective teachers are likely to use terms from everyday language, such as "pinch", "hide", "tie", and similar. Forcing the shift from these situated signs to mathematical signs is just what either the teacher or the teacher educator has to do in mathematical discussion to help learners construct mathematical meaning.

Task 1 aims at making the prospective teachers aware of the features of the given artefact. This is the first step, before being involved in task 2, that influences them to make the artefact evolve on the basis of teaching needs (see chapter 6, this volume).

Consider the "pasta" abacus (see Figure 11), designed by a pair of teachers within a training laboratory, as a variation of the grooved abacus. They had to solve problems met in their classrooms (e.g., the cost of individual teaching aids, the noise, trouble and danger of the falling marbles, the risk that younger kids may swallow beads, and so on). The choice of a very special kind of pasta was carefully discussed: some kinds of macaroni were discarded because a line of macaroni could have hidden the perception of the break between two of them. The size was discussed in order to meet the need of pupils' fingers. When the special shape of "wheels" was chosen, several exemplars were produced for classroom use[7].

The tasks 3, 4 and 5, instead, concern the instrumentation process, and call into play the utilization schemes, that are briefly described below, in another set of tasks. The reader may complete the list for the other artefacts.

Tasks: Artefacts and utilization schemes.

Find the utilization schemes of the counting sticks, the abacus, the pascaline for each of the following tasks: counting and storing data; representing a given number.
Task: *Counting and storing data.*
Counting Sticks
1a (one-to-one correspondence): to move a stick for each item to be counted

1b (counting all): to count all the sticks
2a (bind-tens): to bind with a ribbon or a rubber band ten sticks to obtain a ten-bunch;
2b (bind-hundreds): to bind with a ribbon or a rubber band ten bunches of ten-bunches etc.
Task: *Representing a given number*.
Pascaline
3a (iteration): to repeat the operation of pushing one step clockwise the wheel A on the right until you reach the number.
4a (decomposition) to push clockwise the unit wheel as many steps as needed, the ten wheel as many steps as needed, the hundred wheels as many steps as needed.

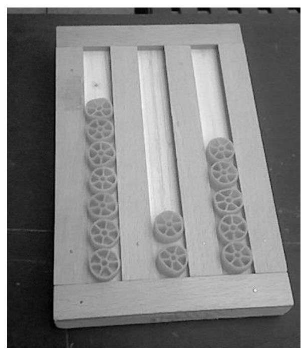

Figure 11. The "pasta" abacus.

For prospective teachers, the careful analysis of arithmetical machines is the first move towards the design of a long term intervention project on the place value of digits at the beginning of primary school.

SECONDARY SCHOOL: SYMMETRY ABOUT AN AXIS IN THE PLANE

The Object to be Mediated

Symmetry is a well known property of those plane figures, which show to be invariant in a mirror image. In geometrical terms, symmetry about an axis is a plane transformation that can be defined as follows:

> Axial symmetry with axis a straight line r is a transformation that for each point P of a plane, defines a point P' such that the line segment PP' is perpendicular to the axis and the midpoint M of PP' is on the axis.

In procedural way, it may also be defined as follows:

> Given a line r (symmetry axis), the image of a point P (not on r) in the symmetry about the axis r is a point P' obtained in the following way. Draw a line n perpendicular to r through P and say H the orthogonal projection of P

on r. Choose the point P’ on n, so that PH = HP’ (being H a point of the line segment PP’). If P is on r, P’=P.

The second definition gives a construction that may be carried out by straightedge and compass.

The Network of Artefacts

Beside the pair “straightedge and compass”, there are other artefacts that allow to construct the symmetrical point P’ of a given point P by means of a direct operation8. In the following, two different artefacts will be illustrated.

The *geodreieck* is a small square (popular in German speaking countries) realized by an isosceles right triangle, where the axis of hypotenuse is traced and the hypotenuse contains two symmetrical number lines with origin in the middle point (Figure 12). Also the other sides of the triangles are graduated in degrees, referring to the inner goniometer. When the axis of the hypotenuse lies on given symmetry axis, two points on the hypotenuse, with the same numerical label, are symmetrical about the symmetry axis.

The *symmetry linkage* is a system of an articulated rhombus and a wooden board with a straight rail; two opposite vertices of the rhombus slide in the straight rail and the other two can carry two pens, in charge of tracing two symmetrical drawing in the plane of the linkage (Figure 13). It dates back to the French scientist C. E. Delaunay (1816–1872).

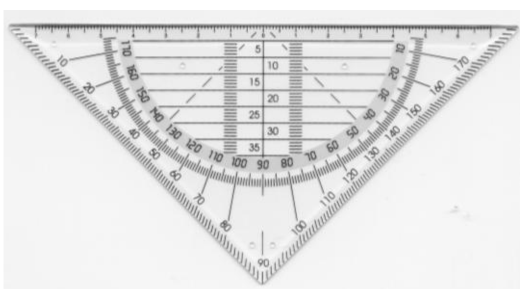

Figure 12. Geodreieck

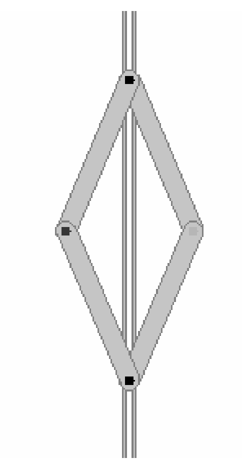

Figure 13. The symmetry linkage[9]

Meanings

All the artefacts (straightedge and compass, geodreieck and linkage) work in a limited part of the plane (see Figure 19 to show the case of linkage). The first definition above, on the contrary, applies to whichever point of the plane. Both geodreieck and linkage evoke the procedural definition, drawing on:

- measuring and coordinate system (geodreieck);
- peculiar properties of rhombuses (in the linkage, diagonals are perpendicular to and bisecting each other).

Instrumentalization

The artefacts are quite different from each other. Geodreieck is a transparent triangle, with printed numbers, referring to the different measures (length and angle width); the linkage has no number at all. Yet a perceptual resemblance appears as also the linkage shows the symmetry axis and some isosceles triangles. Measuring by means of a ruler witnesses that it is a rhombus, within the limit of the measuring tool sensitivity.

Instrumentation

We shall analyse the instrumentation process, starting from some tasks in teacher education.

Tasks in teacher education. A small group of prospective teachers is given the above artefacts together with paper and pencil.
1) *Write the solution of the following task "to draw a pair of symmetrical points about the axis r", using straightedge and compass, the geodreieck and the linkage. Compare the three solutions.*
2) *Write the solution of the following task "to draw two symmetrical triangles about the axis r", using straightedge and compass, the geodreieck and the linkage. Compare the three solutions.*
3) *Write the solution of the following task "to construct the symmetrical triangle A'B'C' of a given triangle ABC about the axis r", using straightedge and compass, the geodreieck and the linkage. Compare the three solutions.*

To draw the symmetrical point of a given point by means of straightedge and compass, one must know the procedural definition of symmetry. On the contrary, some elements of the definition are embodied in the structure of the other artefacts: for instance, the choice of a rhombus for the linkage (although not necessary, see below Figure18) depends on the properties of rhombus. Yet, at the beginning, they are not transparent for the meaning of axial symmetry. With the geodreieck the utilization schemes may be partially guided by an intuitive idea of axial symmetry that may help the reading and the control of points in the two symmetrical number lines. Anyway, it is necessary either to have or to draw the axis on the sheet. It means that the planning of the solution needs a sophisticated consciousness of what is being done, to avoid an improper use of the small square. With the linkage, one must put a sheet on the wooden plate, know the holes where to put the pencil and draw two points; the axis is physical (the rail in the plate). Hence, it is possible to use the artefact by imitation, without being aware of the function of the rail.

To solve the task 2 by means of the geodreieck, the most effective way is to draw 3 pairs of symmetrical points (P, P'; Q, Q'; R, R') and, later, to draw the triangles PQR and P'Q'R'. One must seize the small square with one hand and draw one point at a time with the other as shown in Figure 14. Only at the end, after having drawn the triangles, the symmetry appears evident.

On the contrary, by means of the linkage, one can draw contemporaneously and continuously the two triangles, putting two pens at the two free vertices and piloting them one with the left and the other with the right hand (see Figure 15). If the rail is perpendicular to the subject, as shown in Figure 15, the muscle perception makes clear, during the process, that the two hands are going two symmetrical ways: this is more evident if the user closes the eyes and pays attention to the hand motion only.

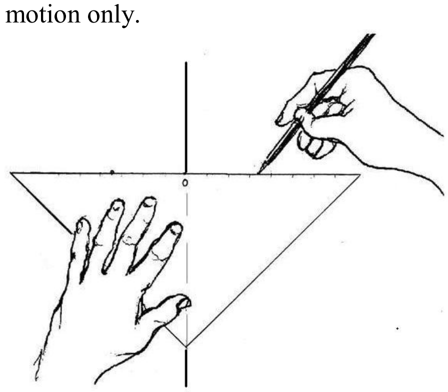

Figure 14. Using a geodreieck.[10]

In the solution of the task 3 (Figure 16) the muscle perception described above disappears and is substituted by the coordination of the gaze (which follows the triangle outline) with the motion of the hand which grips the pencil. This utilization scheme often requires the joint action of two students. The situation is different with the geodreieck.

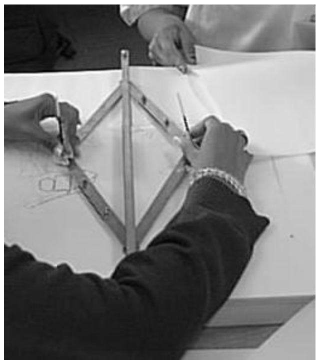

Figure 15. The contemporaneous drawing of two symmetrical figures

Figure 16. The drawing of a figure symmetrical to another one

To solve the task 3, the geodreieck and the linkage confront students with some physical constraints of the artefact (the local feature of the transformation) which induce peculiar utilization schemes. In fact, the given triangle ABC must be drawn close enough to the axis to be accessible for either the hypotenuse of the geodreieck or the free vertex of the rhombus. Only one pencil is sufficient in both cases. One of the free points of the rhombus (pointer) follows the triangle ABC outline, whilst the pencil, placed in the other free point (plotter) draws the triangle A'B'C'.

Generally, in both task 2 and 3, the linkage seems to force students to draw the whole triangle in one motion, although this solution produces always inaccurate drawings (Brousseau, 1986). This is frustrating for many students who would aim to produce precise drawings (as usually required in the geometry lessons). The possibility of improving the precision, by using the linkage to draw only the vertices and by joining them later with a ruler is not usually considered, as if the dynamical feature of the artefact leads necessarily the user towards a global control of the triangle.

Meanings (again)

Until now we have analysed in parallel different artefacts for axial symmetry. In the following we shall analyse in more details different uses of the same artefact (the linkage) at different school levels, then drawing from them some implications for teacher education.

We have introduced the symmetry linkage in the third grade (8 year-olds) of primary school after some experiences with paper folding. In spite of the young age of the pupils, the geometrical structure of the linkage appeared both evident and relevant for meaning construction; the function of the axis and the properties of a rhombus are the main components of the meaning of axial symmetry. Yet a more important experiment was started during the 2006/2007 school year. A long-term teaching project[11] about geometrical transformations was started in grade 7 (12 year-olds (see Figure16). The main focus was on the elaboration of a definition, and this required pupils to draw on the notions of perpendicularity and parallelism.

This experiment was useful also to design the tasks for teacher education. In the laboratory setting of teacher education, prospective teachers are supposed to have learnt some definition of axial symmetry in secondary school: hence, the focus is on the revisitation of the definition and on the justification (*why*) of the linkage functioning. The working sessions are usually split into two parts:

– small group work on the linkage, by means of a working sheet, and
– collective work on the solutions for the given tasks.

Anothe Task in Teacher Education.

A small group of prospective teachers is given the linkage, a working sheet with a drawing (see Figure 17) together with paper and pencil.
Answer the questions, writing carefully your answers.

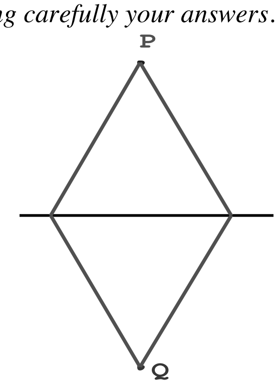

Figure 17. The machine drawing

1. *Describe and represent the machine (How many rigid rods make up the full linkage? Describe the linkage system and measure the lengths of the individual rods…)*
2. *In the machine some vertexes are forced to move in a certain way (bounded vertexes) and other vertexes are free to move on the plane. Which vertexes of the linkage are free? Which ones are bounded? Which are these bounds?*
3. *The two vertexes that are free to move on the plane are called: "pointer" and "plotter". In your machine drawing the pointer is P and the plotter is Q. Put the pencil in the plotter hole and trace the drawing with the pointer* [the point P (the pointer) traces over a given curve or shape and the point Q (the plotter) draws the associated curve or shape (locus) as the linkage moves].

If the pointer (point P) traces out	*the plotter (point Q) draws*
a line segment (of length ……)	

a line segment perpendicular to the straight rail (of length ……)	
a line segment parallel to the straight rail (of length ……)	
a given triangle	

Compare the original figure as followed by the point P and the figure drawn out by the point Q – what do you notice?

4. *When the point P moves along a segment of a given figure, in a particular direction, does the point Q also move along the corresponding side of the plotted figure (locus) in the same direction sense?*
5. *Given a point A, design its correspondent B with the plotter. Explain how the point B can be obtained from the point A (without using the articulated system).*
6. *Try to give a definition of the final transformation produced by this linkage system.*
7. *Choose a Cartesian axes system: if we know the coordinates of the point A, write down the coordinates of the point B in terms of those in A.*
8. *Is it possible to use a different quadrilateral instead of the rhombus?*
9. *What are the shape of the corresponding regions?*
10. *Are there fixed points (i. e. points which have themselves as images)? Are there fixed straight lines (i. e. lines which have themselves as images)?*

Questions 1 and 2 aim at highlighting the physical features of the given artefact (the emergence of the components in the instrumentalisation process) and offer elements to justify the functioning of the linkage. Moreover, question 2 concerns also the instrumentation process (as does question 3), as the manipulation of the linkage is required. Questions 3 and 4, however, aim at highlighting some features of transformation. The straight rail is put in evidence in the question 3 (table), as a reference line. Question 5 rouses the statement of an operative definition (*how*) of axial symmetry, when the linkage is not available any more (but the rail is still in the board). Question 6 asks for a definition.

The following questions, 7, 8, 9, and 10, may be either used in group work or left for the collective discussion. In particular, question 8 prompts a process of conjecture production (*what*) and proof construction (*why*) (Figure 18). It may foster the evolution of the artefact in the instrumentalisation process, by using another linkage to realize the same transformation. Question 9 focuses the limitation of the plane regions where the linkage works (Figure 19).

The limitation of the plane regions is an important constraint. While answering question 3, the students (or prospective teachers) often draw some figures that are too far from the rail and do not succeed in following the figure outline with the pointer. They erase the figure and start again. A new utilization scheme appears: they first look for a region which is close enough to the rail and then draw in that region. The questions about points and lines that have themselves as images are easy to be considered, because the pointer and the plotter may be moved in order to go very close to each other or may be exchanged with each other.

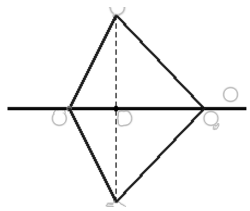

Figure 18. Not only a rhombus

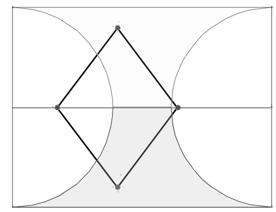

Figure 19. The two corresponding regions

The collective part of the session concerns the shift from the texts produced by the students (or the prospective teachers) towards mathematical texts (definition and properties of symmetry). The different answers (given by different groups) to the question 5 and 6 allow for the start of deeper work on definition. Prospective teachers often bypass these questions and answer only “it is an axial symmetry”. Yet the question is different and concerns the elaboration of a definition. This process starts during group work and ends in the collective discussion, when the comparison between different definitions highlights linguistic issues and tacit assumptions[12]. A typical shift concerns the rail: the initial artefact sign (just “rail”) has to be transformed into “symmetry axis”, that is, a true mathematical sign; the fissure in the wooden board where the linkage is placed becomes, in drawings, a graphical sign with a very precise meaning. It is the only element of the artefact that is maintained in the definition of the transformation as the linkage disappears. Taking away the linkage (question 5) is a strategy to direct students and prospective teachers to focus on the peculiar features of the artefact, i.e., not so much the rhombus as the rail. This is not to be taken for granted. When students and prospective teachers are required to draw the artefact themselves, sometimes they draw only the rhombus (i. e., the most evident part, the element to be handled and moved) and not the rail, which at a first glance may appear non essential.

CONCLUSIONS

In this chapter we have discussed some cases of arithmetical and geometrical machines, showing their use in teacher education, within small size laboratories. The main features of the mathematical machines described in this chapter are the following. The artefacts are to be explored with hands and eyes, hence they exploit the potential of body activity that is at the core of present application of cognitive linguistic and neurosciences to mathematics education (Arzarello and Robutti, in press). The artefacts are taken from the historical phenomenology of mathematics (either arithmetic or geometry), hence they have already shown, in the history, the potential of fostering the construction of mathematical meanings, that have lived through the ages. The reading of historical sources might be placed beside the use of such tools, to make the users aware that they are taking part in historical process that is not individual but collective (Otte and Seeger, 1994).

In the mathematics classrooms, the recourse to physical manipulatives is becoming less frequent and too often substituted with ICT: virtual copies of manipulatives are more and more easily available also for primary school (e. g., http://nlvm.usu.edu/en/nav/index.html). We are not against virtual objects, as a typical task in the Laboratory of Mathematical Machines is the modeling of geometrical machines within a DGE. But we claim that this is only a part of the story and that concrete manipulation has to find a place both in the mathematics classroom and in teacher education. The example of the linkage for symmetry clearly shows some processes (e. g., the transformation of the rail into the axis) that are not expected to emerge in a simulation within DGE, where all the objects are drawn in the same way. ICT are not surrogates for concrete objects; rather they have their own place in mathematics education, because of the features that are partly different from the ones of physical artefacts.

Our genuine interest in ICT is witnessed by another circumstance, the Vygotskian theoretical framework that we have briefly mentioned (Bartolini Bussi and Mariotti, in press) has proved to be effective in the design, implementation and analysis of activities with different kinds of artefacts, i. e., not only classical technologies (as in this chapter, see also Maschietto and Martignone, submitted) but also ICT (Bartolini Bussi and Mariotti, submitted).

As we have highlighted above, the tasks for teacher education have been designed drawing on teaching experiments in primary and secondary school, in order to make prospective teachers capable of planning and running effective classroom activities. The teacher educator's role in the laboratory for prospective teachers is similar to the teacher's role in the classroom, with, at least at the beginning, very similar tasks (and even similar solutions!) in spite of the difference in learners's age. There is, however, a big difference: working with prospective teachers, the process is disclosed and explicitly linked to the theoretical framework and its schematic representation in Figure 7, in order to function as a model of effective classroom activity to be implemented in the teaching profession.

If one assumes the perspective of a teacher educator, it is evident that this increases the complexity of the design of laboratory sessions for prospective teachers compared to laboratory sessions for students. This complexity is consistent with the multidimensional feature of *mathematical knowledge for teaching* (Ball et al., submitted). Actually different domains of knowledge are brought into play, such as:

– the *common content knowledge*, i. e., the mathematical knowledge at stake in the material to be taught (evoked in Figure 20 by the triangle "task – artefact – mathematics knowledge");
– the *knowledge of content and students,* related to the prediction and interpretation of students' processes when a task is given (evoked in Figure 20 by the "cognitive" or "learning" triangle "task – artefact – situated texts");

– the *knowledge of content and teaching*, related to the teacher's actions aiming at the students' construction of mathematical meaning (evoked in Figure 20 by the "teaching" triangle "task – situated texts – mathematical texts");
– the *specialised content knowledge* (evoked in Figure 20 by the "epistemological" triangle "task – mathematics knowledge – mathematical texts").

In each triangle a special function is played by the artefact, i.e., the mathematical machine introduced as tool for teacher education.

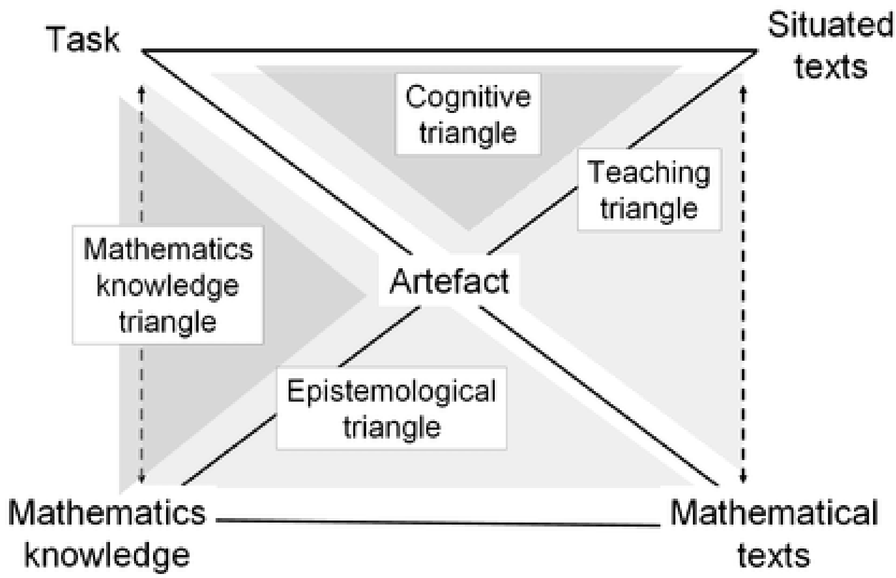

Figure 20. The scheme of Figure 7 revisited.

NOTES

[1] Research funded by MIUR (PRIN 2005019721): Meanings, conjectures, proofs: from basic research in mathematics education to curriculum (national coordinator: M. G. Bartolini Bussi).

[2] In the Italian context this is consistent with the indications of the Mathematics curriculum (see the part on Mathematical Laboratory at http://umi.dm.unibo.it/italiano/Didattica/ICME10.pdf).

[3] An utilization scheme (Rabardel, 1995) is an active structure into which past experiences are incorporated and organized, in such a way that it becomes a reference for interpreting new data. As such, a utilization scheme is a structure with a history, which changes as it is adapted to an expanding range of situations and is contingent upon the meanings attributed to the situations by the individual. This concept allows for the identification of the processes through which an activity is adapted to the diversity of the outside world, in accordance with the particular content to which the scheme is applied

[4] Bèguin and Rabardel (2000) define *instrumentalization as*:

> "We can distinguish several levels of instrumentalization in the attribution of functions to one or more of the artefact's properties. At the first level, instrumentalization is local. It is related to a particular action and to the specific circumstances under which that action occurs. The artefact's properties are given a function temporarily. The artefact is momentarily instrumentalized. At the second level, the artefact's property is more permanently linked to a function that the instrument can perform within a class of actions, objects of the activity, and situations. The instrumentalization is lasting if not permanent. At both of these levels, the artefact itself does not undergo any material transformations. It simply takes on new properties as far as the subject is concerned, acquired either momentarily or more permanently. At the third level, the artefact can be permanently modified in terms of its structure so as to perform a new function" (p. 183).

In the same paper, Bèguin and Rabardel define *instrumentation* as follows:

> "Utilization schemes have both a private and a social dimension. The private dimension is specific to each individual. The social dimension, i.e., the fact that it is shared by many members of a social group, results from the fact that schemes develop during a process involving individuals who are not isolated. Other users as well as the artefact's designers contribute to the elaboration of the scheme" (p. 182).

[5] The text refers to postulates (post.), definitions (def.) and common notions (C.N.), taken from Euclid's Book One.

[6] The pascaline (called "zero+1") is a small (27 cm x 16 cm) plastic tool, produced and sold by the Italian company "Quercetti intelligent toys" (www.quercetti.it).

[7] Two different instrumentalisation processes have been described in this section: the juxtaposition of abaci to represent numbers beyond 9999 and the design of the "pasta" abacus as a variation of the grooved abacus. They are realized by different subjects (pupils vs. teachers). Several differences may be highlighted, with reference to different issues: e. g.,

– task: in the former the pupils are confronted by a mathematical task whilst in the latter the teachers are confronted by a mathematics education task, where all the domains of mathematics knowledge for teaching are at stake (Ball et al., submitted);
– meaning: in the former, the place value as meaning is in progress, whilst in the latter it is taken for granted;
– aim: in the former, the aim is to construct mathematical meaning, whilst in the latter the aim is to produce a tool for constructing mathematical meaning;
– concreteness: in the former, the concrete realization of the artefact must be overcome to represent larger and larger numbers, whilst in the latter the concrete realization is in the foreground;
– relationships: in the former the pupils are faced with a task, an ineffective artefact and a piece of mathematics knowledge, whilst in the latter the teacher are faced with a task, an artefact to be changed, a piece of mathematics knowledge and the pupils who are expected to use the new artefact.

The last issue is related to the different institutional roles played by pupils (who are expected to learn) and teachers (who are expected to teach). It explains why processes and outcomes are expected to be different when even the "same" task

> "Design with everyday materials an arithmetical machine that can be used in the place of the one that has been offered you"

is given to primary school pupils and to primary school teachers.

[8] We shall not consider, in this chapter, paper folding which is, however, a popular way to approach symmetry with young pupils.

[9] The fig. 12 shows a frame of the java simulation of the concrete linkage used by young students in the Figure 14 and in the Figure 15 (http://www.museo.unimo.it/theatrum/macchine/simj/m117.htm: drag the green point to explore). The machines of the MMLab are made of wood and brass. They have different sizes: the large ones (about 70 cm x 50 cm) are used for large group explanations; the

small ones (40 cm x 40 cm) are available in multiple copies for small group work. They can be bought or rented by schools (http://associazioni.monet.modena.it/macmatem/kit%20nuovi.pdf). Individual cardboard or plastic (e. g. geostrips with brass fasteners) models may be cut and assembled in the instrumentalization phase, although, for the instrumentation phase, they show limitations in functioning (they sag and lack holes for pens).

[10] Courtesy of Germana Bartoli.

[11] "Isometric and non-isometric transformations in the plane: a teaching project that makes use of mathematical machines" (Research project carried out by M. Maschietto and F. Martignone).

[12] When linkages concerning less known transformations are into play (e. g., translation), the amount of situated texts in small group work is larger. In these cases the collective work of writing a mathematical definition under the teacher's guide is an important moment of social construction.

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