The Teaching and Learning of Mathematics at University Level

An ICMI Study

Derek Holton (Ed.)

This book arose from the ICMI Study into the teaching and learning of mathematics at university level that began with a conference in Singapore in 1998. The book looks at tertiary mathematics and its teaching from a number of aspects including practice, research, mathematics and other disciplines, technology, assessment, and teacher education. Over 50 authors, all international experts in their field, combined to produce a text that contains the latest in thinking and the best in practice. It therefore provides, in one volume, a state-of-the-art statement on tertiary teaching from a multi-perspective standpoint. No previous book has attempted to take such a wide view of the topic. The book will be of special interest to academic mathematicians, mathematics educators and educational researchers.

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WHAT CAN WE LEARN FROM EDUCATIONAL RESEARCH AT THE UNIVERSITY LEVEL?[1]

1. INTRODUCTION

For more than 20 years, educational research has dealt with mathematics learning and teaching processes at the university level. It has tried to improve our understanding of the difficulties encountered by students and the dysfunction of the educational system; it has also tried to find ways to overcome these problems. What can such research offer to an international study? This is the issue I will address in this article, but first I would like to stress that it is not an easy question to answer, for several reasons including at least the following:

1. Educational research is far from being a unified field. This characteristic was clearly shown in the recent ICMI study entitled "What is research in mathematics education and what are its results?" (See Sierpinska and Kilpatrick, 1996.) The diversity of existing paradigms certainly contributes to the richness of the field but, at the same time, it makes the use and synthesis of research findings more difficult.
2. Learning and teaching processes depend partly on the cultural and social environments in which they develop. Up to a certain point, results obtained are thus time- and space- dependent, their field of validity is necessarily limited. However, these limits are not generally easy to identify.
3. Finally, research-based knowledge is not easily transformed into effective educational policies.

I will come back to this last point later on. Nevertheless, I am convinced that existing research can greatly help us today, if we make its results accessible to a large audience and make the necessary efforts to better link research and practice. I hope that this article will contribute to making this conviction not just a personal one. Before continuing, I would like to point out that the diversity mentioned above does not mean that general tendencies cannot be observed. At the theoretical level, these are indicated, for instance, by the dominating influence of constructivist approaches inspired by Piaget's genetic epistemology, or by the recent move

[1] A shorter version of this paper, Artigue (1999), was published in the *Notices of the American Mathematical Society*.

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attempt to take more account of the social and cultural dimensions of learning and teaching processes (see Sierpinska and Lerman, 1996). But within these general perspectives, researchers have developed a multiplicity of local theoretical frames and methodologies, which differently shape the way research questions are selected and expressed, and the ways they are worked on – thus affecting the kind of results which can be obtained, and the ways they are described. At the cultural level, such general tendencies are also observed. Strong regularities in students' behaviour and difficulties as well as in the teaching problems met by educational institutions, have been observed. These, up to a point, apparently transcend the diversity of cultural environments.

In the following, after characterizing the beginnings of the research enterprise, I will try to overcome some of the above-mentioned difficulties presenting research findings along two main dimensions of learning processes: qualitative changes, reconstructions and breaches on the one hand, cognitive flexibility on the other hand. These dimensions can to some degree, be considered 'transversal' with respect to theoretical and cultural diversities as well as to mathematical domains. No doubt this is a personal choice, induced by my own experience as a university teacher, as a mathematician, and as a education researcher; it shapes the vision I give of research findings, a vision which does not pretend to be objective or exhaustive.

2. FIRST RESEARCH RESULTS: SOME NEGATIVE REPORTS

The first research results obtained at university level can be considered negative ones. Research began by investigating students' knowledge in specific mathematical areas, with particular emphasis on elementary analysis (or calculus in the Anglo-Saxon culture), an area perceived as the main source of failure at the undergraduate level. The results obtained gave statistical evidence of the limitations both of traditional teaching practices and of teaching practices which, reflecting the Bourbaki style, favoured formal and theoretical approaches. The structure and content of the book, *Advanced Mathematical Thinking* (Tall, 1991), gives clear evidence of these facts, noting that:

- by the early eighties, Orton (1980), in his doctoral thesis, showed the reasonable mastery English students had of what can be labelled as 'mere algebraic calculus': calculation of derivatives and primitives (anti-derivatives), but the significant difficulty they had in conceptualizing the limit processes underlying the notions of derivative and integral;
- at about the same time, Tall and Vinner (1981), highlighted the discrepancy between the formal definitions students were able to quote and the criteria they used in order to check properties such as functionality, continuity, derivability. This discrepancy led to the introduction of the notions of concept definition and concept image in order to analyze students' conceptions;

- very early, different authors documented students' difficulties with logical reasoning and proofs, with graphical representations, and especially with connecting analytic and graphical work in flexible ways.

Schoenfeld (1985), also documented the fact that, faced with non-routine tasks, students – even apparently bright students – were unable to efficiently use their mathematical resources.

Research also showed, quite early, that the spontaneous reactions of educational systems to the above-mentioned difficulties were likely to induce vicious circles such as the following. In order to guarantee an acceptable rate of success, an increasingly important issue for political reasons, teachers tended to increase the gap between what was taught and what was assessed. As the content of assessments is considered by students to be what should be learnt, this situation had dramatic effects on their beliefs about mathematics and mathematical activity. This, in turn, did not help them to cope with the complexity of advanced mathematical thinking.

Fortunately, research results are far from being limited to such negative reports. Thanks to an increasing use of qualitative methodologies allowing better explorations of students' thinking and the functioning of didactic institutions (Schoenfeld, 1994), research developed and tested global and local cognitive models. It also organized in coherent structures the many difficulties students encounter with specific mathematical areas, or in the secondary/tertiary transition. It led to research-based teaching designs (or engineering products) which, implemented in experimental environments and progressively refined, were proved to be effective. Without pretending to be exhaustive, let us give some examples, classified according to the two main dimensions given above. (For more details, the reader can refer to the different syntheses in Artigue, 1996, Dorier, 2000, Schoenfeld, 1994, Tall, 1991 and 1996; to the special issues dedicated to advanced mathematical thinking by the journal *Educational Studies in Mathematics* in 1995 edited by Dreyfus; by the journal *Recherches en Didactique des Mathématiques* in 1998 edited by Rogalski; to some of the diverse monographs published by the Mathematical Association of America about calculus reform, innovative teaching practices; and to research about specific undergraduate topics to be found in the MAA Notes on Collegiate Mathematics Education.)

3. QUALITATIVE CHANGES, RECONSTRUCTIONS AND BREACHES IN THE MATHEMATICAL DEVELOPMENT OF KNOWLEDGE AT UNIVERSITY LEVEL

One general and crosscutting finding in mathematics education research is the fact that mathematical learning is a cognitive process that necessarily includes 'discontinuities.' But, depending on the researcher this attention to discontinuities is expressed in different ways. In order to reflect this diversity and the different insights it allows, I will describe three different approaches: the first one, in terms of process/object duality, the second one in terms of epistemological obstacles, the third one in terms of reconstructions of relationships to objects of knowledge.

3.1 *Qualitative changes in the transition from processes to objects: APOS theory*

As mentioned above, research at the university level is the source of theoretical models. The case of APOS theory, initiated by Dubinsky (see Tall 1991) and progressively refined (see Dubinsky and McDonald, this volume, pp. 275-282), is typical. This theory, which is an adaptation of the Piagetian theory of reflective abstraction, aims at modelling the mental constructions used in advanced mathematical learning. It considers that "understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas" Asiala et al, 1996. Of course, this does not occur all at once and objects, once constructed, can be engaged in new processes and so on. Researchers following this theory use it in order to construct genetic decomposition of concepts taught at university level (in calculus, abstract algebra, etc.) and design teaching processes reflecting the genetic structures they have constructed and tested.

As with any model, the APOS model only gives a partial vision of cognitive development in mathematics, but one cannot deny today that it put to the fore a crucial qualitative discontinuity in the relationships students develop with respect to mathematical concepts. This discontinuity is the transition from a process conception to an object one, the complexity of its acquisition and the dramatic effects of its underestimation by standard teaching practices.[2] Research related to APOS theory also gives experimental evidence of the positive role which can be played by programming activities in adequate languages (such as the language ISETL, cf. Tall, 1991) in order to help students encapsulate processes as objects.

Breaches in the development of mathematical knowledge: Epistemological obstacles. The theory of epistemological obstacles, firstly introduced by Bachelard (1938) and imported into educational research by Brousseau (1997), proposes an approach complementary to cognitive evolution, focussing on its necessary breaches. The fundamental principle of this theory is that scientific knowledge is not built in a continuous process but results from the rejection of previous forms of knowledge: the so-called epistemological obstacles. Researchers following this theory hypothesize that some learning difficulties, often the more resistant ones, result from forms of knowledge which are coherent and have been for a time effective in social and/or educational contexts. They also hypothesize that epistemological obstacles have some kind of universality and thus can be traced in the historical development of the corresponding concepts. At the university level,

[2] Note that a very similar approach was developed independently by Sfard, with more emphasis on the dialectic between the operational and structural dimensions of mathematical concepts in mathematical activity (Sfard, 1991).

such an approach has been fruitfully used in research concerning the concept of limit (cf. Artigue 1998 and Tall 1991 for synthetic views). Researchers such as Sierpinska, (1985), Cornu, (1991) and Schneider, (1991) provide us with historical and experimental evidence of the existence of epistemological obstacles, mainly the following:

- the everyday meaning of the word 'limit', which induces resistant conceptions of the limit as a barrier or as the last term of a process, or tends to restrict convergence to monotonic convergence;
- the overgeneralization of properties of finite processes to infinite processes, following the continuity principle stated by Leibniz;
- the strength of a geometry of forms which prevents students from clearly identifying the objects involved in the limit process and their underlying topology. This makes it difficult for students to appreciate the subtle interaction between the numerical and geometrical settings in the limit process.

Let us give one example (taken from Artigue, 1998) of this last resistance, which occurs even in advanced and bright students. In a research project about differential and integral processes, advanced students were asked the following non-standard question: "How can you explain the following: using the classical decomposition of a sphere into small cylinders in order to find its volume and area, one obtains the expected answer for the volume $\frac{4}{3}\pi R^3$, but $\pi^2 R^2$ for the area instead $4\pi R^2$?" It was observed that, faced with this question, the great majority of advanced students tested got stuck. And, even if they were able to make a correct calculation for the area (which they were not always able to do) they remained unable to resolve the conflict.

As the students eventually said, because the pile of cylinders, geometrically, tends towards the sphere, the magnitudes associated with the cylinders behave in the same way and thus have as a limit the corresponding magnitude for the sphere. Such a resistance may look strange but it appears more normal if we consider the effect produced on mathematicians by the famous Schwarz counterexample showing that, for a surface as simple as a cylinder, limits of areas of triangulations when the size of the triangles tends towards 0, can take any value greater than or equal to the area up to infinity, depending on the choices made in the triangulation process, an effect nicely described by in Lebesgue, (1956). The historical and universal commitments of the theory which leads to such results can be discussed and are presently discussed (see, for instance, Radford, 1997). However, what cannot be negated is the fact that the above-mentioned forms of knowledge constitute resistant difficulties for today's students; moreover, that mathematical learning necessarily implies partial rejection of previous forms of knowledge, which is not easy for students.

3.2 Reconstructions in the secondary/tertiary transition: The case of calculus

Qualitative changes in the relationships students develop with respect to mathematical concepts can be approached in a less radical way: in terms of necessary reconstructions. In this section, we illustrate corresponding research findings, by focusing on reconstructions which have been proved to play a crucial role in calculus at the secondary/tertiary transition, at least in the educational situation which tends to predominate now where an intuitive and pragmatic approach to calculus in the secondary curriculum, precedes the formal approach introduced at university. Some of these reconstructions deal with mathematical objects already familiar to students before the official teaching of calculus. Real numbers are a typical example. They enter the secondary curriculum early as algebraic objects with a dense order, with a geometrical representation as the real line, and with decimal approximations that can be easily obtained with pocket calculators. Nevertheless, many pieces of research show that, even upon entering university, students' conceptions remain fuzzy, incoherent, and poorly adapted to the needs of the calculus world. For instance, the ordering of the real numbers is recognized as a dense order. However, depending on the context, students can reconcile this property with the existence of numbers just before or after a given number (0.999... is thus often seen as the predecessor of 1). More than 40% of students entering French universities consider that, if two numbers A and B are closer than $\frac{1}{N}$ for every positive N they are not necessarily equal, just infinitely close. Relationships between irrational numbers and their decimal approximations remain fuzzy. There is no doubt that reconstructions are necessary for understanding 'calculus thinking modes'. Research shows that these are not easily induced by the kind of intuitive and algebraic analysis which is the main focus of calculus instruction at the high school level, and that the constructions of the real number field introduced at the university level have little effect if students are not faced with the incoherence of their conceptions and the resulting cognitive conflicts.

A second category of reconstructions results from the fact that only some facets of a mathematical concept can be introduced at a first contact with it. The concept of integral illustrates this case fairly well. In many countries the first contact with integrals occurs at the upper secondary level via the notion of anti-derivative and a pragmatic approach to the Fundamental Theorem of Calculus which allows the anti-derivative to be connected with an intuitive notion of area. Only at university is a theory of integration developed, first as the theory of Riemann integrals, then, at a more advanced level, as the Lebesgue theory. All of this requires successive reconstructions of the relationships that students have with the integral concept. Much research has been devoted to this theme with great consistency in the results obtained all over the world, documenting the limitations of standard teaching strategies. These results clearly show that reconstruction cannot result from a mere presentation of the theory of Riemann integrals. Through standard teaching practices, students become reasonably successful on standard tasks, but no more. For example, if students are asked in modelling tasks to decide by themselves whether a problem requires an integral process for being solved, they get completely stuck or base their answers on the linguistic hints, if any, that they have learnt to notice in the standard versions of such tasks. Most students think that the safest way to deal successfully with this domain is not to try to understand, but to just function mechanically. I would like to add that we don't have to see this as a sort of cognitive fate. We merely observe our students' economic ways of adaptation to inadequate educational practices.

Research, as was stressed above, is not limited to such negative reports. I would like now to present a situation created by Legrand (1997), in the context of a research project involving mathematicians and physicists with the goal of making first-year university students really feel by themselves the necessity of the integral concept. The situation is based on the following apparently very simple problem (the most effective situations found by researchers are very often apparently simple ones). A linear bar of M_1 and a point mass M_2 are located as shown. Students are asked to calculate the intensity of the attraction between the two masses.

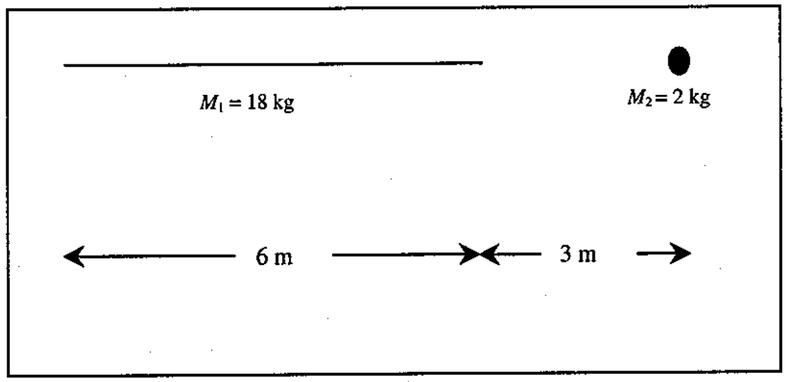

Figure 1. Attraction between a bar and point mass

This situation has been shown effective in various experiments in different contexts. Why is it effective? To answer this question, we need a brief didactic analysis. When asked this question without any linguistic hint, first-year students don't recognize it as an integral problem. But the first important point is that they are not stuck because they can rely on a strategy often used in physics: concentrating the mass of the bar at its centre of gravity and applying the familiar attraction law between two point masses. In experiments, this strategy has always predominated. But, in a group of reasonable size, as is easily the case at university level, there are always students who have some doubts. "Is the gravity principle valid in that particular case?" A second strength of the situation results from the fact that one can test the validity of the gravity principle, simply by applying it in another way. Students generally suggest that the bar be cut into two halves and the gravity principle be applied to each half. Of course, this does not give the same result and the gravity principle is shown to be invalid in that particular case. But this negative answer is also a positive one because it makes salient an essential fact: the contribution of a piece of the bar to the attraction force depends on its distance to the mass x. This allows students to propose upper and lower bounds for the required intensity. Moreover, the technique which was the basis for the invalidation process can be then used in a progressive refinement process, which leads students to the conviction that the force, whose existence is physically attested, can be approximated as accurately as desired. What underlies this is simply the fundamental integral process. In the didactic design elaborated by Legrand, this is just the starting point. Students have then to work on situations that, in different contexts, require the same solution process. Then they have to look for and discuss the analogies between the solutions in order to make the integral process an explicit tool (in the sense of the distinction between the tool and object dimensions of mathematical concepts introduced by Douady, 1987). Only at that point does the university teacher connect this with the theory of Riemann integrals and develop the notion of integral as a mathematical object that will be then reused in more complex situations.

Before leaving this point, let me stress the following: efficiency here is not only linked to the characteristics of the problem which I have just described, it strongly depends on the kind of scenario developed in order to organize students' encounter with this new facet of the integral concept. In a crucial way, this scenario plays on the social character of learning processes. It is through group discussion that the initial strategy is proved to be erroneous. It is the collective game which allows a solution to be found in a reasonable amount of time and which fosters regularities in the dynamic of the situation which could not be ensured if students were faced with the same problem individually or in very small groups. (A similar point is made by Stigler and Hiebert, 1999, p. 164.) No doubt also that the effect would be different if the teacher simply presented this particular example during a lecture.

This example might appear idyllic. But I must confess that educational research does not so easily provide us with effective means to deal with all necessary reconstructions. For instance, differences are evident if one considers the concept of limit, central to calculus. With this particular example, we come to a third category of reconstructions, reconstructions necessary because, as was already acknowledged at the beginning of the last century by the famous mathematician Poincaré (1904), concepts cannot be necessarily taught from the start in their definitive form. At high school level, in most countries today, the impossibility of entering the field of analysis, formally, has been acknowledged. Teaching relies both on a dynamic conception of the limit, based on graphical and numerical explorations, and on techniques of an algebraic nature (Artigue, 1996). These allow students to solve simple but interesting problems of variation and optimization. The transition towards more formal approaches, which takes place at university, represents a tremendous gap both conceptually and technically.

From a conceptual point of view, one crucial point is the following: what is at play through the formalization of the limit concept is, above all, an answer to foundational, unification and generalization needs (see Dorier, 1995, Robert, 1998, or Robert and Speer, this volume, pp. 283-301). It is not easy to make young

students sensitive to such concerns because those concerns are not really part of their mathematical culture. From a technical point of view, the following is essential: in the algebraic analysis of the first contact, technical work does not really break with ordinary algebraic work. This is no longer the case when one enters the field of formal analysis. For example, students must reconstruct the meaning of equality and understand that it doesn't necessarily result, as in algebra, from successive equivalencies, but from ε-proximity for every positive ε. Another point is that inequalities become more frequent than equalities, generating a strong increase in technical complexity, especially so as associated modes of reasoning most often rely on sufficient conditions. These new modes require a carefully controlled loss of information based on a good awareness of the respective orders of magnitude of the different parts of the expressions students have to deal with. In brief, students have a completely new technical world to identify and learn to master. This is far from being easy and is necessarily a long-term process.

3.3 Some concluding remarks: From calculus to linear algebra

Up to now, I have focussed on qualitative changes and more or less radical reconstructions. As stressed above, research shows that teaching practices underestimate both the conceptual and technical costs of these changes. Teaching tends to leave the responsibility for most of the corresponding reorganization to students, with dramatic effects for the majority of these, especially at the secondary/tertiary transition. Research also shows that alternative strategies can be developed fruitfully. Examples have been given for calculus, a domain extensively explored by research. But the growing body of research in linear algebra attests to the existence of similar phenomena (see Dorier and Sierpinska, this volume, pp. 253-272). For instance, the concept of abstract vector space in its axiomatic form, from an epistemological point of view, has been proved to share some common characteristics with the formal concept of limit. When it entered the mathematical scene, its value as a generalizing, unifying, and formalizing concept was stronger than its potential for solving new problems and it was not easily accepted by mathematicians. The same situation occurs with our students who do not need this abstract construction to solve most problems in a first linear algebra course. In France, some researchers have developed specific didactic strategies which aim at making it possible for students to do the necessary reflective and cultural mathematical work (see Dorier et al. 2000). In other countries, these difficulties tend to be removed by reducing topics in first linear algebra courses to those in spaces isomorphic to R^n and by emphasizing matrix calculus and applications Carlson et al (1993). Recent Canadian research (Hillel and Sierpinska, 1994) suggests that this choice is not so benign as it might appear at first sight. Living in a linear algebra world built on the structure of R^n makes it difficult to differentiate vectors and transformations from their canonical representations and can induce further obstacles.

3.4 *Cognitive Flexibility in Learning and Teaching Processes*

The result just mentioned above is linked with a more general issue, that of relationships between mathematical concepts and their semiotic representations, an issue to which educational research pays increasing attention. This fact does not seem independent of the global evolution of theoretical frames mentioned at the beginning of this article, because socio-cultural and anthropological approaches are especially sensitive to the role played by the material and symbolic tools of mathematical activity in learning processes. Depending on the theoretical perspective, this attention is expressed in different ways, but the fundamental point is that it breaks with a common vision of instrumental and semiotic competencies as a by-product of conceptualization and hypothesizes strong dialectic relationships in their mutual development. This is of particular importance, especially if one has in mind the current technological evolution of the instruments of mathematical activity. More generally, mathematical learning can no longer be seen, as is often the case, only as a regular ascension towards higher levels of abstraction and formalization. Connections between mathematical fields of experience, points of view, settings, and semiotic registers are a crucial part. With such considerations in mind, we enter a wider domain that could be labelled the domain of cognitive flexibility, which is increasingly investigated by research (see, for instance, Dreyfus and Eisenberg, 1996).

I will use some examples taken from recent research in linear algebra in order to illustrate this point. As stressed by Dorier (2000), historically linear algebra helped to unify different pre-existing mathematical settings: geometry, linear systems in finite and infinite dimensions and determinants, differential equations, and functional analysis. This unifying role and power is an essential epistemological value of linear algebra that has to be understood and used by students. But this cannot be achieved without the development of complex connections among reasoning modes, points of view, languages, and systems of symbolic representations. Once more, research helps our understanding of the complexity of the necessary cognitive constructions and, at the same time, shows the insensitivity of the educational system to this complexity. In Dorier (2000), for instance, on the one hand, Hillel points out the necessary interaction in linear algebra between three different levels of language and representations: those of the general theory, of geometry, and of R^n. On the other hand, Sierpinska et al. show the necessary interaction between three different reasoning modes, respectively labelled as synthetic and geometric, analytic and arithmetic, analytic and structural.[3] Both show the inadequacy of the different teaching practices documented, from lectures to tutorials. Alves Dias (1998), in her recent doctoral thesis, analyses the relationships between two fundamental points of view in linear algebra: the parametric and Cartesian points of view.[4] She clearly shows that, even if the conversion between parametric and Cartesian representations of vector subspaces is, a priori, easily achieved thanks to ordinary techniques for solving systems of linear equations, when dealing with vector spaces of finite dimensions, a flexible connection between these two points of view is far from being mastered by advanced French and Brazilian students. Mathematical symbols such as matrices can foster errors in the use of those formal representations because students operate on the formal symbols without checking to see if the operations they perform are meaningful in terms of the objects the symbols represent. This often leads to absurd results which are not recognised by students because they do not interpret or check their findings through geometrical or dimensional arguments. The detailed analysis of textbooks Alves Dias carried out, shows that they don't pay attention to these questions or develop theoretical arguments, for instance in terms of duality, which remain too far away from the technical level to make students able to control the connection.

These are examples in linear algebra. As documented by research, mutatis mutandis, there are similar examples in calculus. In that more extensively explored area, research also provides experimental evidence that computer technologies, if properly used (which is not so easy) can play a crucial role in fostering flexible connections among semiotic representations. For instance, among graphical, numerical, and symbolic representations of functions, and help graphical representations to become effective tools of the mathematical work (see Tall, 1991 and Dubinsky and Harel, 1992). Research also shows that the effective use of computer technologies requires the development of specific mathematical knowledge, a requirement which is not easily accepted by an educational institution whose values have been traditionally defined with respect to paper and pencil environments.

4. POTENTIAL AND LIMITS OF RESEARCH FOR ACTION ON THE EDUCATIONAL SYSTEM

As we have tried to show in this article, research carried out at the university level helps us better understand the learning difficulties our students have to face, the surprising resistance of some, and the limitations and dysfunction of some of our teaching practices. Moreover, in various cases, research has led to the production of teaching designs that have been proved to be effective, at least in experimental environments. But we must also recognize that research does not give us a general way to easily improve the learning and teaching processes. Some reasons can be found in the current state of research: up to now, efforts have been concentrated on a few domains taught at university level. Also, the training of future mathematicians at the expense of the great diversity of students taking university mathematics courses, has more or less implicitly been assumed. Research remains thus very partial due

[3] In the synthetic mode, mathematical objects are, in some way, directly given to the mind, which tries to grasp and describe them. In the analytic mode, they are given indirectly: built through definitions and properties of their elements. This analytic mode is divided by researchers into two different sub-modes: the analytic-arithmetic where objects are given by a formula which makes it possible to calculate them, and the analytic-structural where objects are defined by a set of properties.

[4] A parametric point of view is adopted with a vector subspace for instance if the subspace is characterized by some set of generators. A Cartesian point of view is to characterize a subspace as the solutions of a linear system or as the null space of a linear operator.

both to the content it explores and to its vision of the expected form and content of knowledge. In my opinion, the way the issue of computer technologies has been generally addressed evidences this fact. It mainly focuses on the ways computer technologies can support conceptualization and the cognitive flexibility recognized as an essential component of this conceptualization. It does not give the same attention to what is really a professional mathematical activity assisted by computer technologies, and the specific and non-specific mathematical needs, depending on professional specialty, required to become an efficient and critical user and how the corresponding knowledge can be constructed in ordinary or service mathematics courses. Nevertheless, this is also a real challenge we must face today, taking into account the fact that, at university, our main concern is no longer the development of some kind of general mathematical culture.

Other reasons such as the following seem more fundamental: it is rare that research allows us to think that through minimal and cheap adaptations we could obtain substantial gains. On the contrary, most research-based designs require more engagement and expertise from teachers, and significant changes in practices (see for instance Dubinsky, Mathews and Reynolds, 1997 as regards collaborative learning). One essential reason is this. What has to be reorganized is not only the content of teaching (it is not enough to write or adopt new textbooks), but more global issues such as the forms of students' work, the modes of interaction between teachers and students, and the forms and contents of assessment. This is not easy to achieve and is not just a matter of personal good will.

Another crucial point is the complexity of the systems where learning and teaching take place. Because of this complexity, the knowledge that we can infer from educational research is necessarily very partial. The models we can elaborate are necessarily simplistic ones. We can learn a lot even from simplistic models but we cannot expect that they will give us the means to really control didactic systems. So we must be realistic in our expectations and careful about generalizations. This does not mean, in my opinion, that the world of research and the world of practice must live and develop separately. Far from it. But it does mean that finding ways of making research-based knowledge useful outside the communities and experimental settings where it develops cannot be left as the sole responsibility of researchers. It is our common task.

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PURPOSES AND METHODS OF RESEARCH IN MATHEMATICS EDUCATION[1]

Bertrand Russell has defined mathematics as the science in which we never know what we are talking about or whether what we are saying is true. Mathematics has been shown to apply widely in many other scientific fields. Hence, most other scientists do not know what they are talking about or whether what they are saying is true.

Joel Cohen, *On the nature of mathematical proofs*

There are no proofs in mathematics education.

Henry Pollak

1. INTRODUCTION

The first quotation above is humorous, the second serious. Both, however, serve to highlight some of the major differences between mathematics and mathematics education – differences that must be understood if one is to understand the nature of methods and results in mathematics education.

The Cohen quotation does point to some serious aspects of mathematics. In describing various geometries, for example, we start with undefined terms. Then, following the rules of logic, we prove that if certain things are true, other results must follow. On the one hand, the terms are undefined – i.e., "we never know what we are talking about." On the other hand, the results are definitive. As Gertrude Stein might have said, a proof is a proof is a proof.

Other disciplines work in other ways. Pollak's statement was not meant as a dismissal of mathematics education, but as a pointer to the fact that the nature of evidence and argument in mathematics education is quite unlike the nature of evidence and argument in mathematics. Indeed, the kinds of questions one can ask (and expect to be able to answer) in educational research are not the kinds of questions that mathematicians might expect. Beyond that, mathematicians and education researchers tend to have different views of the purposes and goals of research in mathematics education.

This paper begins with an attempt to lay out some of the relevant perspectives, and to provide background regarding the nature of inquiry within mathematics

[1] A closely related paper (Schoenfeld, 2000a) was published in the *Notices* of the American Mathematical Society.

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